## Question 1: One loop divergences of scalar Yang-Mills

Write down the Lagrangian for a Yang-Mills gauge field $A_{\mu}$ minimally coupled to a complex scalar field $\Phi$ in the fundamental representation, and use it to find the vertices and propagators at tree level. Do not choose any particular gauge group. Then choose any oneloop diagram that you can build using your Feynman rules, and write down the expression for it as an integral over loop momentum $k$. Estimate how this one-loop diagram will blow up in the UV (at high $k$ ) but do not try to evaluate it explicitly.

## Question 2: Large- $N$ 't Hooft expansion

Sometimes it is convenient to use the matrix representation of the Yang-Mills gauge field $A_{\mu}$. The propagator behaves like

$$
\begin{aligned}
\langle 0| T\left\{A_{\mu}(x)^{I}{ }_{J} A_{\nu}(0)_{L}^{K}\right\}|0\rangle & =\langle 0| T\left\{A_{\mu}^{A}(x) A_{\nu}^{B}(0)\right\}|0\rangle\left(T^{A}\right)^{I}{ }_{J}\left(T^{B}\right)^{K}{ }_{L} \\
& \propto \delta^{A B}\left(T^{A}\right)^{I}{ }_{J}\left(T^{B}\right)^{K}{ }_{L} \quad \propto \delta^{I}{ }_{L} \delta^{K}
\end{aligned}
$$

This motivates the introduction of double line notation, in which the gauge information in the Yang-Mills field is represented in Feynman graphs by (oriented) double lines rather than single wavy lines. The propagator and vertices take the form


Motivated by the difficulty of solving QCD, suppose that we take the rank of the gauge group $N$ to become parametrically large (without altering matter couplings). The hope is that $1 / N$ will provide a perturbative handle on the physics. So we take

$$
N \rightarrow \text { large }, \quad \lambda \equiv g_{\mathrm{YM}}^{2} N=\text { fixed }
$$

where $\lambda$ is known as the 't Hooft coupling.
Draw several examples of possible loop diagrams. Notice how some of them are planar, in the sense that you can draw them on the paper without lifting your pen. Others are non-planar. Figure out how many powers of $\lambda$ and $1 / N$ are attached to each type of graph, just by using group theory data and the topology of the Feynman graph. Then, from the patterns you see, argue that loop corrections in the Feynman graph expansion are organized in a double perturbation expansion in powers of $\lambda$ and powers of $1 / N$. (Do not try to prove the double expansion rigorously.)

## Question 3: $R_{\xi}$ gauges

Consider an Abelian gauge field $A_{\mu}$ coupled to a complex scalar $\phi=\frac{1}{\sqrt{2}}\left(\phi^{1}+i \phi^{2}\right)$ with

$$
\mathscr{L}=-\frac{1}{4} F^{\mu \nu} F_{\mu \nu}-\left|D_{\mu} \phi\right|^{2}-V(\phi)
$$

(a) Write down the infinitesimal form of the local $U(1)$ gauge symmetry.
(b) Take the potential $V(\phi)$ to have the form we encountered earlier with the Mexican Hat potential. Suppose that $U(1)$ spontaneously breaks, with $\phi$ developing a vev along (say) the $\phi^{1}$ direction. Expand

$$
\phi^{1}(x)=v+h(x), \quad \phi^{2}(x)=\varphi(x) .
$$

Choose the following gauge-fixing function (instead of, say, the Lorentz gauge) for this $U(1)$ symmetry:

$$
G=\frac{1}{\sqrt{\xi}}\left(\partial_{\mu} A^{\mu}-\xi \operatorname{ev} \varphi\right)
$$

Using this specific form of $G$, derive the Feynman rules (propagators and vertices) for all fields. Show in particular that although the ghost and antighost decouple from the gauge field they do not decouple from the Higgs.

Note: A generalization of this $R_{\xi}$ gauge to the non-Abelian gauge theory of the Standard Model of particle physics is used to prove renormalizability. See Peskin \& Schröder chapter 21 or Weinberg chapter 17 (volume II) for more details.

Question 4*: Supersymmetry [NOT COMPULSORY; FOR EXTRA CREDIT ONLY]
Supersymmetry is a special type of symmetry with a fermionic parameter that connects bosonic and fermionic fields. Its generators $Q_{\alpha}$ obey an algebra known as the super-Poincaré algebra: $\left\{Q_{\alpha}, Q_{\beta}\right\}=2 \gamma_{\alpha \beta}^{\mu} P_{\mu},\left[P^{\mu}, Q_{\alpha}\right]=0,\left[M^{\mu \nu}, Q_{\alpha}\right]=\left(\frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right]\right)_{\alpha}^{\beta} Q_{\beta}$, but this will not concern us here.

Consider the free Wess-Zumino action for a complex scalar $\phi$ and a Weyl fermion $\chi$,

$$
\mathscr{L}_{\mathrm{WZ}}^{\mathrm{free}}=\left(\partial^{\mu} \phi^{*} \partial_{\mu} \phi+i \chi^{\dagger} \bar{\sigma}^{\rho} \partial_{\rho} \chi+F^{*} F\right)-m\left[\phi F+\frac{i}{2} \chi^{T} \sigma^{2} \chi+\text { c.c. }\right]
$$

(a) Use the equations of motion to show that $\phi$ and $\chi$ have the same mass. (Hint: you may need to eliminate the auxiliary field $F$ via its equation of motion to see this.) Does the number of degrees of freedom in the bosonic sector of this model match the number of degrees of freedom in the fermionic sector, both on-shell and off-shell? Explain.
(b) Show that the kinetic part of the above action is invariant under SUSY transformations of the form

$$
\begin{gathered}
\delta \phi=-i \epsilon^{T} \sigma^{2} \chi \\
\delta \chi=\epsilon F+\sigma^{\rho} \partial_{\rho} \phi \sigma^{2} \epsilon^{*} \\
\delta F=-i \epsilon^{\dagger} \bar{\sigma}^{\rho} \partial_{\rho} \chi
\end{gathered}
$$

where the supersymmetry parameter $\epsilon$ is a 2-component spinor of Grassmann numbers.
(c) Show that the mass term is also invariant under SUSY transformations.
(d) Show that the following specific interaction Lagrangian also respects SUSY:

$$
\mathscr{L}_{\mathrm{WZ}}^{\mathrm{int}}=\left\{F \frac{\partial W[\phi]}{\partial \phi}+\frac{i}{2} \frac{\partial^{2} W[\phi]}{\partial \phi^{2}} \chi^{T} \sigma^{2} \chi+\text { c.c. }\right\}
$$

where $W$ is an arbitrary function of $\phi$ known as the superpotential. For the case $W=g \phi^{3} / 3$, write out the field equations for $\phi$ and $\chi$ (after eliminating $F$ ).

